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Algorithm Theory - Winter Term 2017/2018
Exercise Sheet 5

Hand in by Thursday 10:15, January 11, 2017

Exercise 1: Maximal Matching (4+6 Points)

Consider the following simple algorithm to find a maximal matching in a given graph $G = (V, E)$. Consider an initially empty set M. Pick an arbitrary edge $\{u, v\} \in E$ and add it to M. Then, remove all the edges adjacent to u or v from E. Repeat adding edges from E to M, as explained, until E becomes empty.

(a) Show that the algorithm computes a matching of size at least half the size of an optimal matching.

Now, let us assume that each edge e in the given graph $G = (V, E)$ is assigned a positive integer w_e as its weight.

(b) Provide a greedy algorithm (by adapting the above algorithm) to find a maximal matching with weight at least half of the weight of an optimal matching. Show why the solution is within factor 2 of an optimal solution.

Exercise 2: Perfect Matching (5 Points)

For a positive integer r, an r-regular graph is a graph where each node has the same degree r. Show that any r-regular bipartite graph has a perfect matching.

Exercise 3: Ford Fulkerson Revisited (10 Points)

Show that the below statement is correct or prove that it does not hold.
Often the Ford Fulkerson algorithm needs to consider many augmenting paths. If the algorithm always chooses the 'correct' augmenting paths it never has to choose more than $|E|$ paths.

Exercise 4: Large Chromatic Number without Cliques (1+5+5+3+1 Points)

A c-coloring of a graph $G = (V, E)$ is a function $\phi : V \to \{1, \ldots, c\}$ such that any two neighboring nodes have different colors, i.e., for each $\{u, v\} \in E$, $\phi(u) \neq \phi(v)$. The chromatic number $\chi(G)$ of a graph G is the smallest integer c such that a c-coloring of G exists, e.g., the chromatic number of a k-node clique is k. In the following we use probability theory to show that not only cliques imply large chromatic number, in particular we would like to show the following:

For any k and l there is a graph with chromatic number greater than k and no cycle shorter than l.

In the following consider a (random) graph $G_{n,p}$ on n nodes, where each (possible) edge $\{u, v\}, u, v \in V$ exists with probability $p = n^{\frac{1}{2l}-1}$.

(a) An independent set I of a graph G is a set of nodes such that no two nodes in I are neighbors in G. The independence number $\alpha(G)$ of a graph denotes the size of the largest independent set.

Explain why $\chi(G) \geq |V(G)|/\alpha(G)$ holds.

(b) Show that for $a = \lceil \frac{3}{p} \ln n \rceil$ we have

$$\Pr[\alpha(G) \geq a] \longrightarrow_{n \to \infty} 0.$$

Hint: There are $\binom{n}{a}$ choices for the nodes of an independent set of size a. What is the probability that a specific set of nodes of size a form an independent set? Also use the linearity of expectation!

(c) Let X be the number of cycles of length at most l. For large n, show that $E[X]$ can be upper bounded by $\frac{n}{4}$.

Hint: What is the probability that j specific nodes form a cycle? How many choices of nodes that can possibly form a cycle of length less than l are there? Again, use the linearity of expectation.

(d) From (b) and (c), we can deduce that $\Pr[X \geq n/2$ or $\alpha(G) \geq a] < 1$ holds. This means that there exists a graph H with n nodes where the number of cycles with length less than l is less than $n/2$ and the independence number is smaller than a. So H has a small independence number but it might contain some short cycles.

Explain how to modify the graph H to obtain a graph H' with no cycles of length at most l, $\alpha(H') < a$ and $|V(H')| \geq n/2$.

(e) Show that the graph H' has no cycle of length at most l and a chromatic number at least k.

Remark: All subquestions in this exercise can be solved independently from each other (by using the results of the other questions as black box).